partial tr_{ace}

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1 Partial Trace

Informally, it is easy to find the partial trace for some forms. For a given system ρ_{AB} in the form $|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|$ we define the partial trace on B as follows:

$$\rho_A = tr_B(\rho_{AB}) = tr_B(|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \equiv (|a_1\rangle\langle a_2|) \cdot tr(|b_1\rangle\langle b_2|)$$

if we have tri-partite state in the form $\rho_{ABC} = |a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2| \otimes |c_1\rangle\langle c_2|$ and we wish to trace out C:

$$\rho_{AB} = tr_C(\rho_{ABC}) = (|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \cdot tr(|c_1\rangle\langle c_2|)$$
$$\Leftrightarrow tr_C(\rho_{ABC}) = (|a_1\rangle\langle a_2| \otimes |b_1\rangle\langle b_2|) \cdot \langle c_1|c_2\rangle$$

Example: Let $\rho_{AB} = |\psi_{AB}\rangle\langle\psi_{AB}|$ with $|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$
Now if we take the partial trace: $\rho_A = tr_B(\rho_{AB})$ we get :

$$\begin{aligned}\rho_{AB} &= \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|) \\ &= \frac{1}{2}[(|00\rangle + |11\rangle)(\langle 00| + \langle 11|)] \\ &= \frac{1}{2}[|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|]\end{aligned}$$

If we want to trace B out to get the reduced density matrix on A, we need to take the trace over the subspaces of the Hilbert space for the second qubit. Simply put, we want to trace B (the second system) out so we should look at the second qubit here represented in bold.

$$\rho_A = tr_B(\frac{1}{2}[|0\mathbf{0}\rangle\langle 0\mathbf{0}| + |0\mathbf{0}\rangle\langle 1\mathbf{1}| + |1\mathbf{1}\rangle\langle 0\mathbf{0}| + |1\mathbf{1}\rangle\langle 1\mathbf{1}|)$$

We only keep the first half that represents A for each expression, we take all the terms that have $|0\rangle\langle 0|$ or $|1\rangle\langle 1|$ for the second qubit.(i.e both RHS and LHS are equal (because these will be elements on the diagonal so the trace will be 1, alternatively you can look at their scalar product $\langle 0|0\rangle$ and $\langle 1|1\rangle$)

$$\rho_A = tr_B(\frac{1}{2}[\underline{|0\mathbf{0}\rangle\langle 0\mathbf{0}|} + \cancel{|0\mathbf{0}\rangle\langle 1\mathbf{1}|}) + \cancel{|1\mathbf{1}\rangle\langle 0\mathbf{0}|} + \underline{|1\mathbf{1}\rangle\langle 1\mathbf{1}|}))$$

We get the following state :

$$\rho_A = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|) = \frac{\mathbb{1}}{2}$$

Which is maximally mixed state, since it can be written as $\frac{\mathbb{1}}{d}$ (with $\mathbb{1}$ being the identity matrix and d the dimension of our system, here 2) also you can clearly see that $tr(\rho_A^2) = 0.5$

Reminder:

- $tr(\rho_A^2) = 1 \Rightarrow$ pure state
- $\frac{\mathbb{1}}{d} < tr(\rho_A^2) < 1 \Rightarrow$ mixed state
- $tr(\rho_A^2) = \frac{\mathbb{1}}{d} \Rightarrow$ maximally mixed state